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TRANSIENT TEMPERATURE DISTRIBUTIONS IN CYLINDRICAL SHELLS

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Abstract—Transient temperature distributions in a cylindrical shell are calculated for the case that heat is transferred at a constant rate to the inner wall surface, while the outer wall surface is insulated against heat losses. Temperature profiles for nine Fourier numbers and three different thickness–diameter ratios are given.

The truncation errors introduced by taking only the first root of the characteristic equation are calculated for three situations. Also the error introduced when the quasistationary profile is taken instead of the total solution of the differential equation is calculated for three situations.

NOMENCLATURE

- k, heat conductivity $[\mathrm{MLt^3T^{-1}}]$;
- α, thermal diffusivity $[\mathrm{L^2t^{-1}}]$;
- Q, heat flux per unit length in axial direction $[\mathrm{MLt^{-3}}]$;
- t, time [t];
- T, temperature [T];
- T_0, initial temperature [T];
- R_i, radius of the inner wall [L];
- R_0, radius of the outer wall [L];
- Θ, dimensionless temperature, $2\pi k(T - T_0)/Q$;
- R, dimensionless radial coordinate, r/R_i;
- Fo, dimensionless time (Fourier), $\alpha t/R_i^2$;
- λ, ratio between outer wall radius and inner wall radius, R_0/R_i.

DERIVATION OF THE SOLUTION TO THE PROBLEM

We consider a cylindrical wall with inner radius R_i and outer radius R_0. At the inner wall surface a constant flux per unit length (Q) is transferred to the wall. The outer wall surface is insulated. The initial temperature is uniform throughout the wall. This situation can arise, e.g. in tunnel walls of atomic shelters [1].

The unsteady-state heat balance in cylindrical coordinates reads:

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right). \tag{1}$$

The initial and boundary conditions for the present problem are:

$$\left.\begin{aligned} T &= T_0, & t &= 0,\\ \frac{\partial T}{\partial r} &= 0, & r &= R_0,\\ \frac{\partial T}{\partial r} &= -\frac{Q}{2\pi k R_i}, & r &= R_i. \end{aligned}\right\} \tag{2}$$

Introducing the dimensionless variables as indicated in the Nomenclature (1) and (2) become:

$$\frac{\partial \Theta}{\partial Fo} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Theta}{\partial R}\right) \tag{3}$$

$$\left.\begin{aligned} \Theta &= 0, & Fo &= 0,\\ \frac{\partial \Theta}{\partial R} &= 0, & R &= 1,\\ \frac{\partial \Theta}{\partial R} &= -1, & R &= \lambda. \end{aligned}\right\} \tag{4}$$

We will present the general solution obtained by means of the Laplace transformation. Carslaw and Jaeger [2] indicate the outlines of a procedure for solution of (3). They give a general solution for linear boundary conditions but the conditions (4) are excluded (p. 332, footnote).

Applying the Laplace transformation to (3) we obtain for $\mathscr{L}\{\Theta\} \equiv \overline{\Theta}$:

$$\overline{\Theta}(R, s) = \frac{(\sqrt{s})K_1(\lambda\sqrt{s})I_0(R\sqrt{s}) + (\sqrt{s})I_1(\lambda\sqrt{s})K_0(R\sqrt{s})}{s^2[I_1(\lambda\sqrt{s})K_1(\sqrt{s}) - I_1(\sqrt{s})K_1(\lambda\sqrt{s})}. \tag{5}$$

Inversion to the (R, Fo) domain of this formula offers no particular difficulties. The double pole at $s = 0$ gives rise to a term linear in Fo after inversion.

Applying the complex inversion integral on (5) by using the Heaviside inversion theorem (Abramowitz and Stegun [3] p. 1021), we obtain after some computations:

$$\Theta(R, Fo) = \frac{\lambda^2}{\lambda^2 - 1}\ln\left(\frac{\lambda}{R}\right) + \frac{\lambda^2}{(\lambda^2 - 1)^2}\ln\lambda + - \frac{\lambda^2 - R^2}{2(\lambda^2 - 1)} - \frac{\lambda^2 + 1}{4(\lambda^2 - 1)} + \frac{2}{\lambda^2 - 1}.Fo + -\pi\sum_{n=1}^{\infty} C_n(\lambda)\left[J_0(\mu_n R)\,Y_1(\mu_n\lambda) - J_1(\mu_n\lambda)\,Y_0(\mu_n R)\right]\exp(-\mu_n^2 Fo) \tag{6}$$

where:

$$C_n(\lambda) = \frac{J_1(\mu_n)\,J_1(\mu_n\lambda)}{\mu_n[J_1^2(\mu_n\lambda) - J_1^2(\mu_n)]}. \tag{7}$$

The μ_n are the positive roots of the characteristic equation:

$$J_1(\mu\lambda)\,Y_1(\mu) - J_1(\mu)\,Y_1(\mu\lambda) = 0. \tag{8}$$

The first five roots of (8) are given in [3], p. 415, for ten values of λ. In our calculations we used the first fifty roots of (8) for three values of λ: (5/2, 5/3, 5/4).

Figures 1–3 give the transient temperature distributions for eight Fourier times. One sees that for high values for Fo the temperature rises linear with time throughout the wall. This quasi-steady-state profile can readily be found from a heat balance over the wall. Inspection of equation (6) of course also gives this profile if one recognizes that the contribution of the last term will be damped out exponentially with time.

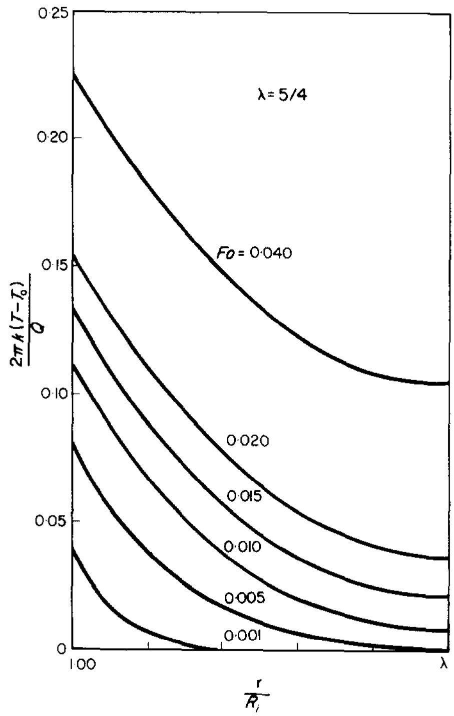

FIG. 1. Temperature distribution in a shell with thickness of 0·25 the internal radius.

For practical calculations one is interested in the errors introduced when the infinite summation is truncated. We inspected two situations, first when the series is truncated after five terms and second when only the first root of the characteristic equation is used. The errors were indicated with Δ_5 and Δ_1 respectively. The errors were defined as:

$$\Delta_5 = \left|\frac{\Theta_{50\text{ terms}} - \Theta_{5\text{ terms}}}{\Theta_{50\text{ terms}}}\right|.100$$

and

$$\Delta_1 = \left|\frac{\Theta_{50\text{ terms}} - \Theta_{1\text{ term}}}{\Theta_{50\text{ terms}}}\right|.100.$$

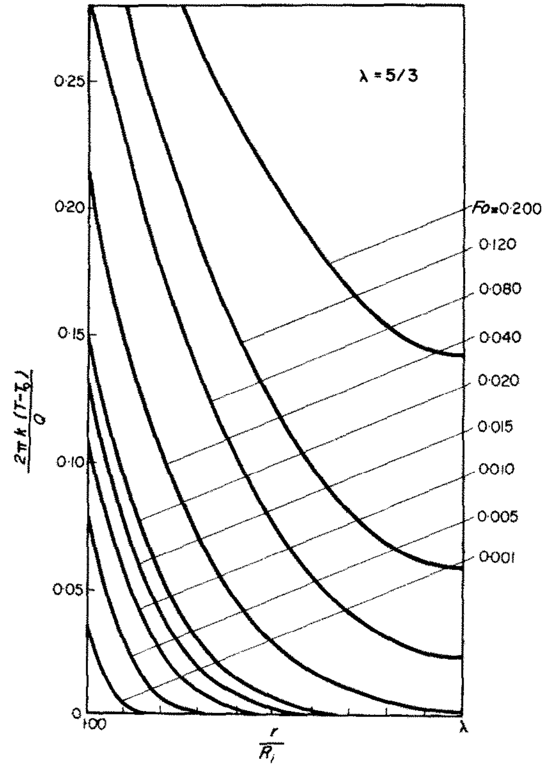

FIG. 2. Temperature distribution in a shell with thickness of 0·66 the internal radius.

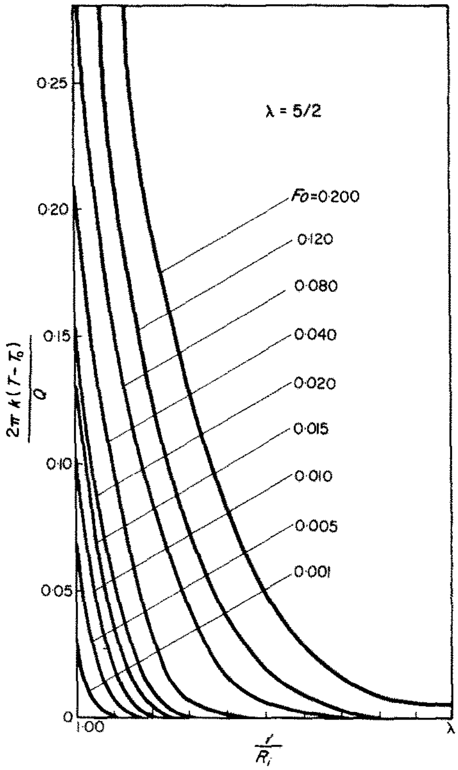

FIG. 3. Temperature distribution in a shell with thickness of 1·50 the internal radius.

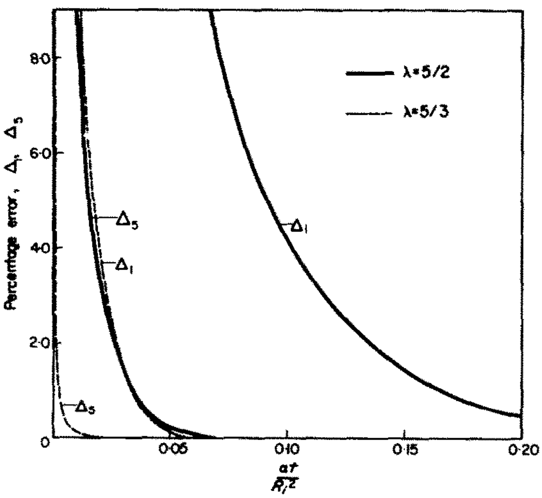

FIG. 4. Truncation error for one term (Δ_1) and five terms (Δ_5) of the series in equation (6) for two values of λ.

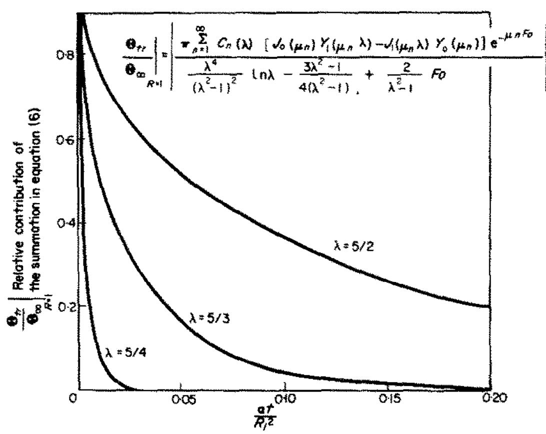

FIG. 5. Relative contribution of the summation in (6) to the final result as a function of Fo with three parametric λ values.

In Fig. 4 the results are summarized. It can be seen that for values of $Fo > 0{\cdot}20$ the temperature profiles are determined with sufficient accuracy by taking only the first term. For $Fo > 0{\cdot}05$ the first five eigenvalues are sufficient. The accuracy of the truncated series increases with decreasing λ.

For very large values of Fo the whole influence of the infinite summation can be neglected in comparison with the first part of formula (6). In Fig. 5 the ratio between the contribution of the summation and the quasi-stationary profile are sketched. The summation can be neglected without serious loss of accuracy for $Fo > 0{\cdot}2$ when $\lambda = 5/3$. In Figs. 4 and 5 the temperature at the inner wall was chosen.

REFERENCES

1. C. D. Scott and J. S. Newman, Heat storage in tunnel walls, *I/EC Fundamentals* **6**, 3 (1967).
2. H. S. Carslaw and J. C. Jaeger, *Heat Conduction In Solids*, 2nd edn. Clarendon Press, Oxford (1959).
3. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover, New York (1965).

Résumé—Les distributions de température transitoires dans une coque cylindrique sont calculées dans le cas où la chaleur est transportée à une vitesse constante à la surface de la paroi intérieure, tandis que la surface de la paroi extérieure est isolée contre les pertes thermiques. On donne les profils de température pour neuf nombres de Fourier et trois rapports différents de l'épaisseur sur le diamètre.

Les erreurs de troncatures introduites en prenant seulement la première racine de l'équation caractéristique sont calculées dans trois cas. On calcule également dans trois cas l'erreur introduite lorsqu'on prend le profil quasistationnaire au lieu de la solution complète de l'équation différentielle.

Zusammenfassung—Instationäre Temperaturverteilungen in einer zylindrischen Zelle werden für den Fall berechnet, dass eine konstante Wärmemenge zur inneren Zelloberfläche geleitet wird, während die äussere Oberfläche gegen Wärmeverlust isoliert ist. Temperaturprofile werden für neun Fourier-Zahlen und drei verschiedene Verhältnisse von Dicke zu Durchmesser angegeben.

Für drei Fälle sind die Abrundungsfehler infolge der Verwendung nur der ersten Wurzel der charakteristischen Gleichung berechnet. Auch der durch die Einführung des quasistationären Profiles anstelle der totalen Lösung der Differentialgleichung auftretende Fehler wird für drei Fälle berechnet.

Аннотация—Рассчитываются распределения температуры в цилиндрической оболочке в переходной зоне для случая постоянного теплового потока к внутренней стенке и теплоизолированной внешней стенки. Приводятся температурные профили для девяти чисел Фурье и для трех различных относительных толщин.

Для трех случаев оцениваются погрешности, вызываемые учетом только первого корня характеристического уравнения. Для трех случаев оценивается погрешность, когда используется квазистационарный профиль вместо общего решения дифференциального уравнения.